Catania, August 22, 2017

Prof. Elena Guardo
Dipartimento di Matematica e Informatica
University of Catania – Italy

Doctoral Thesis Report on
"The effect of points fattening on del Pezzo surfaces"
by mgr Magdalena Lampa–Baczyńska

A del Pezzo surface (first studied by del Pezzo in 1887 in [5]) is a smooth projective algebraic variety of dimension two whose anticanonical divisor $-K_X$ is ample. The degree d of a del Pezzo surface X is by definition the self intersection number $d = (K_X)^2$ of its canonical divisor K_X.

Over an algebraically closed field, every del Pezzo surface of degree d is either a product of two projective lines (with $d = 8$), or the blow-up of a projective plane in $r = 9 - d$ points in general position, i.e, with no three collinear, no six on a conic, and no eight of them on a cubic having a node at one of them. Conversely, any blowup of the plane in points satisfying these conditions is a del Pezzo surface.

In particular, over the complex field $\mathbb{C}$ there are exactly ten del Pezzo surfaces: $\mathbb{P}_2, \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{S}_r$, for $1 \leq r \leq 8$, that are obtained by blowing-up $\mathbb{P}^2$ in $1 \leq r \leq 8$ points in general position. We denote these blow ups by $f_r : \mathbb{S}_r \to \mathbb{P}_2$. We have $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$, and $K_{\mathbb{P}^1\times\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2)$. Even if the anticanonical bundle has an intrinsic definition, in terms of an exceptional configuration, for all the other del Pezzo surface if E_i denotes the class of the 1-dimensional scheme-theoretic fiber $f_r^{-1}(p_i)$ of $\mathbb{S}_r$ over the point p_i and H denotes the pullback to $\mathbb{S}_r$ of the class of a line in $\mathbb{P}^2$, it is

$$\mathbb{L}_r = -K_{\mathbb{S}_r} = 3H - E_1 - \ldots - E_r$$

which is not divisible in the Picard group $Pic(\mathbb{S}_r)$. We also note the inclusions between the linear systems

$$|\mathbb{L}_8| \subset |\mathbb{L}_7| \subset \ldots \subset |\mathbb{L}_1|. \tag{1}$$

For the convenience of the reader, we recall some definition about the fattening following [6]. Let X be a smooth projective variety and let L be an ample bundle on X and let Z be a reduced subscheme of X defined by the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_Z$. For a positive integer m one defines

1. the initial degree of the subscheme mZ with respect to L as the integer $\alpha(mZ) := min\{d : H^0(X, dL \otimes \mathcal{I}_Z^{(m)}) \neq 0\}$

2. the initial sequence with respect to L

$$\alpha(Z), \alpha(2Z), \ldots,$$

3. the Waldschmidt constant of the subscheme Z with respect to the bundle L as the limit

$$\hat{\alpha}(Z) := \lim_{m\to\infty} \frac{\alpha(mZ)}{m}$$

It is proved that this limit exists and it is equal to $inf\{\frac{\alpha(mZ)}{m} : m \in \mathbb{N}\}$ (see [4], Lemma 2.3.1, or [9] Remark III.7. Alternatively, use Feketes Lemma [8] as in [1]).

The alpha problem is an interesting topic to study, and many authors (Baczyńska, Bocci, Chiantini, Di Rocco, Dumnicki, Harbourne,Lundman, Malara, Pokora, Szemberg, Szpond, Tutaj-Gasińska, just to cite some of them) extended the study to other spaces.

Bocci and Chiantini started the study of the effect of points fattening in $\mathbb{P}^2$ in [3]; in particular, in Theorem 3.3, they proved that there does not exist any zero-dimensional subscheme Z of $\mathbb{P}^2$, with two equal consecutive numbers of the initial sequence and the smallest difference between the consecutive numbers is 1. Moreover, the zero-dimensional subschemes $Z \subset \mathbb{P}^2$ such that $\alpha(2Z) - \alpha(Z) = 1$ are either contained in a line or form a star configuration.

The results of Bocci and Chiantini were generalized by Dumnicki, Szemberg, Tutaj-Gasińska in [7], where they study configurations of points in $\mathbb{P}^2$ with $\alpha((m+1)Z) - \alpha(mZ) = 1$ for some $m \geq 2$ and obtained their full characterization. So, it is natural to ask:

(Q1) what is the minimal growth of the initial sequence (i.e., the minimal difference between its two consecutive elements),

(Q2) under which geometric conditions the initial sequence of a subscheme attains the minimal growth.

This kind of questions can be studied using fairly classical tools from algebraic geometry and commutative algebra. In this thesis, the author gives many interesting results on effect of points fattening and detailed pictures only for the surfaces $\mathbb{S}_r$ for $1 \leq r \leq 8$. The projective plane and $\mathbb{P}^1 \times \mathbb{P}^1$ are not studied in the dissertation.

The case $\mathbb{P}^2$ was studied in [3] and [7]. Here, we can use geometrical interpretation of symbolic powers due to Nagata-Zariski (see [2], Theorem 2.2).

The case $\mathbb{P}^1 \times \mathbb{P}^1$ was studied in [1]. On this surface it is natural to work with α-invariant taken with respect to the line bundle $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$. The main result characterizes the sets Z of points in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $\alpha((m+1)Z) = \alpha(mZ)$ for some $m \geq 1$, i.e., iff Z is a grid in $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover, they also gave Chudnovsky-type bounds, i.e, if Z is a set of reduced points in $\mathbb{P}^1 \times \mathbb{P}^1$, then $\frac{\alpha(mZ)}{m} \geq \frac{\alpha(Z)}{2}$ for all $m \geq 1$.

The aim of this project is also to complete the picture for all remaining del Pezzo surfaces $\mathbb{S}_r$ for $r = 1, \dots, 8$.

The surface $\mathbb{S}_1$ is studied in the last chapter (Chapter 6) of the thesis. Recall, that $\mathbb{S}_1$ arises from the blow-up of the projective plane in a fixed point. This surface, is both a del Pezzo surface and a Hirzebruch surface since $\mathbb{S}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. From the point of view of Hirzebruch surfaces the reference line bundle for $\mathbb{S}_1$ is $2H - E_1$, while considered as del Pezzo surface, the anticanonical bundle is $\mathbb{L}_1 = 3H - E_1$.

The effect of points fattening on $\mathbb{S}_1$ as Hirzebruch surface, was studied by Di Rocco, Lundman and Szemberg in [6]. In particular, in Proposition 4.1, they proved that there does not exist any finite set Z such that

$$\alpha(Z) = \alpha(2z) = \alpha(3Z) = \alpha(4Z).$$

They also were able to give again a characterization of configurations of points in $\mathbb{S}_1$ with minimal growth rate of the initial sequence and to prove a Chudnovsky-type inequality for the α-invariant computed with respect to $-K_{\mathbb{S}_1}$.

In this thesis, Theorem 6.1, the author proved that if we considered $\mathbb{S}_1$ as del Pezzo surface, then there exist sets Z satisfying the condition

$$\alpha(Z) = \alpha(2z) = \alpha(3Z) = \alpha(4Z) = \alpha(5Z).$$

Lemma 6.2 and Corollary 6.3 concern the Waldschmidt constant in the following cases

1. if $Z = \{Q\}$ and Q and $Q \in E_1$, then $\hat{\alpha(Z)} = \frac{1}{5}$
2. if $Z \subset \mathbb{S}_1$ is any finite set of points, then $\hat{\alpha(Z)} \geq \frac{1}{5}$.

The del Pezzo surface $\mathbb{S}_2$ is studied in Chapter 2. Recall that it is the blow-up of the projective plane in two points P_1 and P_2 and the reference line bundle is $\mathbb{L}_2 = 3H - E_1 - E_2$. In this case, the author gives answers and all natural questions related to (Q1) and (Q2).

Theorem 2.1 characterizes the sets Z on $\mathbb{S}_2$ such that

$$\alpha(Z) = \alpha(2z) = \alpha(3Z) = \alpha(4Z) = \alpha(5Z).$$

The author gives also a complete description of the behaviour of initial sequences for various sets of finite points on $\mathbb{S}_2$. Corollary 2.8 shows a strong lower bound for the constant $\hat{\alpha}(Z)$.

Chapter 3 is devoted to the del Pezzo surface $\mathbb{S}_3$. Theorem 3.1 characterizes finite sets of points for which

$$\alpha(Z) = \alpha(2z) = \alpha(3Z) = \alpha(4Z) = 1.$$

The author proposed a conjecture to give a full characterization of sets satisfying the equality $\alpha(mZ) = \ldots = \alpha((m+3)Z)$ (see Conjecture 3.6). Although Theorem 3.9 characterizes five types of sets Z such that $\alpha(Z) = \alpha(2Z) =$

$\alpha(3Z) = 1$, there exist many other sets where $\alpha(mZ) = \alpha((m+1)Z) = \alpha((m+2)Z)$ for some integer m (see Theorem 3.14 for a family of such sets). There are also some results regarding the computation of $\hat{\alpha}(Z)$.

The surfaces $\mathbb{S}_4, \mathbb{S}_5$, and $\mathbb{S}_6$ are studied in Chapter 4. In particular, Theorem 4.1, Theorem 4.3 and Theorem 4.6 give the complete description of finite sets of points $Z \subset \mathbb{S}_r$ for $r = 4, 5, 6$ such that

$$\alpha(Z) = \alpha(2Z) = \alpha(3Z) = 1.$$

Also, the author gives results for $\hat{\alpha}(Z)$.

Chapter 5 is devoted to the surfaces $\mathbb{S}_7$ and $\mathbb{S}_8$. Here, the author gives a characterization of sets $Z \subset \mathbb{S}_7, \mathbb{S}_8$ such that $\alpha(Z) = \alpha(2Z) = 1$ (see Theorem 5.1 and Theorem 5.4) and results on $\hat{\alpha}(Z)$.

Notice that the line bundle $\mathbb{L}_8$ has the least number of sections of all line bundles $\mathbb{L}_r$. For this reason, it was expected that $\alpha(2Z) \geq 2$, while surprisingly in Theorem 5.4 the author proves the existence of a set $Z \subset \mathbb{S}_8$ for which $\alpha(Z) = \alpha(2Z) = 1$.

It should be also remarked that the line bundles $\mathbb{L}_7$ and $\mathbb{L}_8$ are not very ample and Lemma 1.25 of this thesis does not work. Example 5.7 shows that if we apply the lemma to the bundle $\mathbb{L}_8$ on $\mathbb{S}_8$ we get $\frac{\alpha(mZ)}{m} \geq 1$ whereas Lemma 5.6 gives an example with $\hat{\alpha}(Z) = \frac{1}{2}$.

Chapter 6 is divided in three sections: the first is devoted to $\mathbb{S}_1$ (as we described before). In Section 6.2 the author presents Chudnovsky-type results for surfaces $\mathbb{S}_r$ for $r \leq 6$ (Theorem 6.13).

Finally, in Section 6.3, the author gives an overview of its main results. It is worth noting that for $\mathbb{S}_1$ and $\mathbb{S}_2$ we can obtain up to five equal consecutive values in the initial sequence. Probably, it is the maximal length of constant initial sequence for all the del Pezzo surfaces.

Because of the sequence (1), the reference bundle on $\mathbb{S}_r$ has less sections than the reference bundle on $\mathbb{S}_{r-1}$. So, it is natural to expect that a sequence of equalities

$$\alpha(Z) = \alpha(2Z) = \ldots = \alpha(mZ) = 1$$

should be shorter with growth of r for the surface $\mathbb{S}_r$. We can summarize the obtained results as the following: for $r = 1, 2$ we have $m = 5$ and it is not possible to obtain more than five consecutive initial values equal; for $r = 3$ we have $m = 4$, for $r = 4, 5, 6$ we have $m = 3$ and for $r = 7, 8$ we have $m = 2$.

The author ends with an interesting open problem. She asks if these values of m are also maximal values, i.e., condition

$$\alpha(aZ) = \alpha((a+1)Z) = \ldots = \alpha((a+m-1)Z)$$

can be satisfied for any integer a. This is true for $\mathbb{S}_1$ and $\mathbb{S}_2$, but the tools used by the author cannot be adapted for surfaces $\mathbb{S}_r$ with $r \geq 3$. Such a problem is still an open one.

Conclusion The dissertation is organized in a very good way.

All prerequisites, specific to the subject studied here, are clearly explained and typically illustrated with carefully chosen, illuminating examples. Some examples are studied in more and more advanced settings throughout the thesis. I think, this is a very good idea to accompany the theoretical settings by examples which are familiar to the reader already from earlier chapters. I think, in this way, one can appreciate the accumulating knowledge and growing selection of tools applicable in very concrete situations.

The technical level of this presentation is very good and adequate to its target audience. The subject is very interesting and it gives new ideas for future works.

I strongly recommend the acceptance of this doctoral thesis.

Prof. Elena Guardo

References

[1] M. Baczyńska, M. Dumnicki, A. Habura, G. Malara, P. Pokora, T. Szemberg, J. Szpond, H. Tutaj-Gasińska, *Points fattening on* $\mathbb{P}^1 \times \mathbb{P}^1$ *and symbolic powers of bi-homogeneous ideals*, J. Pure Appl. Alg. 281 (2014), 1555–1562.

[2] M. Baczyńska, G. Malara, *On the containment hierachy for simplicial ideals*, J. Pure Apll. Alg. 219 (2015), 5402-5412.

[3] C. Bocci, L. Chiantini, *The effect of points fattening on postulation*, J. Pure and Appl. Alg. 215 (2011), 89–98.

[4] C. Bocci, B. Harbourne, *Comparison Powers and Symbolic Powers of Ideals*, J. Algebraic Geometry, 19 (2010), 399– 417.

[5] P. del Pezzo, *Sulle superficie dell' n-esimo ordine immerse nello spazio a n dimensioni*, Rend. Circ. Mat. Palermo 12 (1887), 241-271.

[6] S. Di Rocco, A. Lundman, T. Szemberg, *The effect of points fattening on Hirzebruch surfaces*, Math. Nachr. 288 (2015), 577–583.

[7] M. Dumnicki, T. Szemberg, H. Tutaj-Gasińska,*Symbolic powers of planar point configurations*, J. Pure Appl. Alg. 217 (2013), 1026–1036

[8] H. Fekete, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichung en mit ganzzahligen Koeffizienten.* Math. Z. 17 (1923), no. 1, 228-249.

[9] B. Harbourne, J. Roé,*Computing multi-point Seshadri constants on* $\mathbb{P}^2$. Bull. Belg. Math. Soc. Simon Stevin 16 (2009), no. 5, 887-906.