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Referee report on the PhD thesis by mgr Magdalena Lampa-Baczyńska
"The effect of points fattening on del Pezzo surfaces"

The PhD thesis by Ms. Magdalena Lampa-Baczyńska is devoted to the so-called effect of points fattening on del Pezzo surfaces. Let us recall that a smooth, geometrically irreducible and proper surface over a field $\mathbb{K}$ is called a **del Pezzo surface** iff its anticanonical divisor is ample (i.e. $(-\mathrm{K}_X)$ is ample). Moreover, it is well-known that for $\mathbb{K} = \mathbb{C}$ there are exactly ten del Pezzo surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the surfaces $\mathbb{S}_r$ that are obtained as blow-ups $f_r : \mathbb{S}_r \to \mathbb{P}^2$ with center consisting of $r \in \{1, \ldots, 8\}$ generic points $P_1, \ldots, P_r$ on the projective plane $\mathbb{P}^2$. The anticanonical bundle of $\mathbb{S}_r$ is given by the formula

$$\mathbb{L}_r := 3H - E_1 - \ldots - E_r,$$

where $E_1, \ldots, E_r$ are the exceptional divisors of the blow-up f_r.

In order to discuss the result of the thesis under review we have to recall the notion of the initial degree. Let X be a smooth projective variety and let L be an ample bundle on X. For a positive integer m and a reduced subscheme $Z \subset X$ given by the ideal sheaf $\mathcal{I}_Z$ one defines the **initial degree** of (mZ) with respect to L as the integer

$$\alpha(mZ) := \min\{d \,:\, \mathrm{H}^0(X, dL \otimes \mathcal{I}_Z^{(m)}) \neq 0\}\,.$$

By varying the integer $m \in \mathbb{N}$ one obtains the weakly growing, subadditive sequence of integers (the so-called **initial sequence** of the subscheme Z with respect to the bundle L):

$$(\alpha(mZ))_{m=1}^{\infty}$$

This in turn leads to the notion of the **Waldschmidt constant** of the subscheme Z with respect to the bundle L:

$$\hat{\alpha}(Z) := \lim_{m\to\infty} \frac{\alpha(mZ)}{m} = \min\{\frac{\alpha(mZ)}{m} \,:\, m \in \mathbb{N}\}\,,$$

where the second equality follows from the classical Fekete's Lemma. Finally, it should be mentioned that the notions we just discussed are of interest because they are closely related to various classical problems of algebraic gemetry (see e.g. the papers by Ch. Bocci and L. Chiantini, B. Harbourne, T. Szemberg).

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The thesis under review consists of Introduction and six chapters.

Introduction contains concise exposition of the current state of knowledge on the so-called alpha problem. In particular, the few results that are discussed in the introduction allow the reader to understand the motivation of Ms. Magdalena Lampa-Baczyńska to study the problems that are the main subject of the thesis under review. It is well-known that given a zero-dimensional subscheme Z of $\mathbb{P}^2$, its initial sequence with respect to $\mathcal{O}_{\mathbb{P}_2}(1)$ is strictly increasing, whereas the zero-dimensional subschemes $Z \subset \mathbb{P}^2$ such that $\alpha(2Z) - \alpha(Z) = 1$ are either contained in a line or form a star configuration (Bocci and Chiantini, JPAA 2011). Thus it is natural to ask:

($\mathbf{Q}_1$) what is the minimal growth of the initial sequence (i.e., the minimal difference between its two consecutive elements),

($\mathbf{Q}_2$) under which geometric conditions the initial sequence of a subscheme attains the minimal growth.

In the first chapter "Preliminaries" the author introduces the basic notation and discusses certain well-known facts that play important role in her considerations. The chapter is quite short but it does help the reader to understand the thesis.

Chapter 2 is devoted to the del Pezzo surface $\mathbb{S}_2$. The author fixes the bundle $L = \mathbb{L}_2$, and answers all natural questions related to (Q1), (Q2). For the convenience of the reader we quote some of the results of this chapter:
By Theorem 2.1 in the thesis, a zero-dimensional reduced subscheme $Z \subset \mathbb{S}_2$ satisfies the condition

$$\alpha(Z) = \ldots = \alpha(5Z) = 1$$

iff it is a subscheme od the intersection $\tilde{L}_{12} \cap (E_1 \cup E_2)$, where $\tilde{L}_{12}$ is the proper transform of the line through the centers P_1, P_2. Moreover, the integer $\alpha(6Z)$ exceeds 1 for every zero-dimensional reduced $Z \subset \mathbb{S}_2$ (Corollary 2.2). Exactly four initial values of the initial sequence are minimal iff Z is a point in the exceptional locus of f_2 but away from the curve $\tilde{L}_{12}$ (see Theorem 2.3). Six consecutive values of the initial sequence never coincide for a finite set of points $Z \subset \mathbb{S}_2$ by Theorem 2.7. All finite sets of points on $\mathbb{S}_2$ such that five values of the initial sequence coincide are described in Theorem 2.11. The finite sets of points $Z \subset \mathbb{S}_2$ on the exceptional divisors, such that their initial sequence has stability index four are characterized in Theorem 2.13. Theorem 2.20 is a geometric characterization of finite sets of points on $\mathbb{S}_2$ for which the first three values of the initial sequence are minimal.
The complete understanding of behaviour of initial sequences for various sets of finite points on $\mathbb{S}_2$ allows Ms. Lampa-Baczyńska to compute the Waldschmidt constant $\hat{\alpha}(Z)$ for various subschemes $Z \subset \mathbb{S}_2$ with respect to the bundle $\mathbb{L}_2$ (see e.g. Corollary 2.10), and prove a strong lower bound for the constant $\hat{\alpha}(Z)$ (c.f. Corollary 2.8, Lemma 1.25).

In Chapter 3 Ms. Lampa-Baczyńska turns her attention to the surface $\mathbb{S}_3$ with the bundle $L = \mathbb{L}_3$. Theorem 3.1 in the thesis characterizes finite sets of points for which four initial values of the initial sequence are minimal, whereas the zero-dimensional reduced subschemes of $\mathbb{S}_3$ with exactly three minimal initial values of $(\alpha(mZ))_{m=1}^{\infty}$ are described in Theorem 3.9. The geometry of $\mathbb{S}_3$ is far more complex than the one of $\mathbb{S}_2$, so the author does not prove an analogue of Theorem 2.11 for $\mathbb{S}_3$ and four consecutive values of the initial sequence, but she is able to obtain some partial results in this direction (Example 3.2, Example 3.3, Lemma 3.5) and formulate Conjecture 3.6, that gives a hypothetical picture in the case of the del Pezzo surface $\mathbb{S}_3$. Chapter 3 contains also explicit computations of (resp. bounds for) $\hat{\alpha}(Z)$ for some finite sets of points $Z \subset \mathbb{S}_3$. It should be pointed out that the existence of the curves introduced on $\mathbb{S}_3$ by the third blow-up considerably increases the combinatorial complexity of the problems related to (Q1), (Q2) in this case. Thus the description of the behaviour of initial sequences in Chapter 3 is less complete than the one in Chapter 2, but the arguments are more interesting and involved (see e.g. the proof of Theorem 3.14).

Chapter 4 contains the complete description of finite sets of points $Z \subset \mathbb{S}_r$ such that the three initial values of the initial sequence are minimal for $r = 4, 5, 6$ (see Theorem 4.1, Theorem 4.3 and Theorem 4.6). The author also examines the behaviour of the Waldschmidt constants (see e.g. Corollary 4.7). We skip a more detailed discussion of those results to maintain our exposition compact.

In Chapter 5 Ms. Lampa-Baczyńska examines the surface $\mathbb{S}_r$ with the bundle $\mathbb{L}_r$ for $r = 7, 8$. It should be pointed out that the bundles in question are no longer very ample for those values of the parameter r. The characterizations of finite subsets of $\mathbb{S}_r$, where $r = 7, 8$, with two minimal initial values of $(\alpha(mZ))_{m=1}^{\infty}$ can be found in Theorem 5.1 and Theorem 5.4. In particular, quite unexpectedly the author shows that such subschemes exist on $\mathbb{S}_8$ (recall that $\mathrm{h}^0(\mathbb{L}_8) = 1$). Chapter 5 also contains some results on the Waldschmidt constants in this case.

Finally, the first section of the last chapter of the thesis is devoted to the surface $\mathbb{S}_1$, that is also the Hirzebruch surface $\Sigma_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1))$. The effect of points fattening on Hirzebruch surfaces was examined by the thesis supervisor together with Di Rocco and Lundman (Math. Nachr. 288 (2015)). In the case of Σ_1 the three authors studied the behaviour of the bundle $(2H - E_1)$ to show that even four initial values of the initial sequence with respect to the bundle $(2H - E_1)$ cannot coincide. In contrast, Ms. Lampa-Baczyńska shows that all results she proved for the pair $(\mathbb{S}_2, \mathbb{L}_2)$ in Chapter 1 of her thesis remain valid (after some natural modifications) for the pair $(\mathbb{S}_1, \mathbb{L}_1)$. I find this section very interesting, because it provides some insight on the behaviour of the initial sequence when we vary the reference bundle. The second section of Chapter 6 is devoted to the proof of a lower bound on the growth rate of the initial sequence of a finite set of points on the surface $\mathbb{S}_r$ for $r \leq 6$ (Theorem 6.13). Finally, the section 6.3 is a concise overview of the results of the thesis.

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To sum up, the thesis of mgr Magda Lampa-Baczyńska deals with problems that are subject of intensive research. The presented results are original, interesting and could be useful for other researchers. The proofs presented in the thesis show that the author posesses sound geometric intuition, a good command of classical techniques of complex algebraic geometry (especially the theory of algebraic surfaces) and certain amount of creativity. One cannot avoid the impression that the author of the thesis devoted much effort to simplify the proofs. The presentation of the results in the thesis confirms Ms. Lampa-Baczyńska's ability to explain her ideas in a precise and efficient manner.
Conclusion: In my opinion, the thesis "The effect of points fattening on del Pezzo surfaces" submitted by mgr Magdalena Lampa-Baczyńska fulfills all requirements posed on a PhD thesis. Consequently, I recommend it for the defense in front of the appropriate committee.

Conclusion in Polish: Moim zdaniem praca spełnia wymogi dotyczące rozprawy doktorskiej zawarte w ustawie o tytule naukowym i stopniach naukowych. Wnioskuję o dopuszczenie Pani magister Magdaleny Lampa-Baczyńskiej do dalszych etapów przewodu doktorskiego.

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